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ON THE STABILITY OF A FREELY INTERACTING BOUNDARY LAYER*

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There is considered the stability of a plane-parallel boundary layer on a plate relative to three-dimensional longwave perturbations. The Prandtl equations are used to describe their development in time, where the pressure gradient is assumed to be induced by internal wave interaction with the external potential flow. For an incompressible fluid the approach mentioned yields results that follow from ordinary linear stability theory under the condition that the critical layer of neutral oscillations adjoins the streamlined surface. For a subsonic velocity at infinity the dispersion relation for the spatial perturbations reduces to the standard form, that gives the parameters in a nonstationary two-dimensional free interaction process.

1. Viscous sublayer. For any (either sub- or supersonic) velocity at infinity, free boundary layer interaction is subject to the Prandtl equations for an incompressible fluid in the near-wall domain /1—4/

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\quad \frac{\partial p}{\partial y}=0 \tag{1.1}$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial y^2}$$

Here a special dimensionless system of reference units is used in which t denotes the time, x, y and z are Cartesian coordinate axes, u, v and w are velocity vector projections on the designated axes, and p is the excess pressure. In classical boundary-layer theory, the pressure is found in advance by solving the external problem about the flow over a given body by an inviscid gas. The free interaction process is characterized by the fact that the pressure refers to a number of required functions, and is determined by combining the solution for the near-wall layer with the solution that yields the external potential flow parameters. The combine conditions will be formulated later.

For simplicity, we consider the streamlined body to be a flat plate. The adhesion conditions on its surface $y=0$ are

$$u=v=w=0 \tag{1.2}$$

We seek the solution of (1.1) in the form

$$u=y-a\frac{df}{dy}\exp(\omega t+kx+lz),\quad v=ag\exp(\omega t+kx+lz) \tag{1.3}$$
$$w=-a\frac{dh}{dy}\exp(\omega t+kx+lz),\quad p=a\exp(\omega t+kx+lz)$$

and we linearize the resulting relationships with respect to the perturbation amplitude a. Formulas (1.3) are traditional in stability theory /5,6/; they permit going from partial to ordinary differential equations for the functions f, g and h. Integrating one of them yields the final relation

$$g=kf+lh$$

which can be used to eliminate g. The other two functions f and h satisfy the equations

$$\frac{d^3f}{dy^3}-(\omega+ky)\frac{df}{dy}+kf+lh+k=0 \tag{1.4}$$
$$\frac{d^3h}{dy^3}-(\omega+ky)\frac{dh}{dy}+l=0$$

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On the basis of an idea by Squire /5,6/, we multiply the first one by k, the second by l and we add the relationships obtained in this manner. We consequently arrive at one differential equation for a new required function F

$$\frac{d^3F}{dy^3}-(\omega+ky)\frac{dF}{dy}+kF=0,\quad F=\frac{k}{k^2+l^2}(kf+lh) \tag{1.5}$$

Let us note that the function f satisfies exactly the same equation if the parameters of the freely interacting boundary layer are independent of z. On the plane of the complex variable Z the equation (1.5) reduces to an Airey equation

$$\frac{d^4F}{dZ^4}-Z\frac{d^2F}{dZ^2}=0,\quad Z=\frac{\omega}{k^{2/3}}+k^{1/3}y \tag{1.6}$$

for d^2F/dZ^2. In order to extract the regular branch of the three-valued function $k^{1/3}$, we make a slit in the k plane from the origin to infinity along the negative real half-axis. We set $\arg k^{1/3}=0$ on the positive real half-axis. Then the inequalities $-\pi/3\leqslant\arg k^{1/3}\leqslant\pi/3$ will be valid.

The adhesion conditions (1.2) permit formulation of boundary conditions for the function F. Namely, we have $F=dF/dZ=0$ for $Z=\omega/k^{2/3}$. Eliminating the solution of (1.6) that grows exponentially at infinity, and complying with the boundary conditions mentioned, we find

$$F=-\left[\frac{d\,\mathrm{Ai}(\zeta)}{dZ}\right]^{-1}\int\limits_{\zeta}^{Z}dZ'\int\limits_{\zeta}^{Z'}\mathrm{Ai}(Z'')\,dZ'',\quad \zeta=\frac{\omega}{k^{2/3}} \tag{1.7}$$

where $Ai(Z)$ is the Airey function. In order to set up a relation between the frequency ω and the wave numbers k and l, we turn to the external flow domain in which the influence of viscosity and heat conduction can be neglected.

2. Potential flow. Here the excess pressure satisfies the equation

$$\mp\frac{\partial^2p'}{\partial x^2}+\frac{\partial^2p'}{\partial y'^2}+|M_\infty^2-1|^{-1}\frac{\partial^2p'}{\partial z^2}=0 \tag{2.1}$$

in which there are no time derivatives /3,4/. The length measurement scales along the x, z axes and the pressure p' in the external domain remain the same as in the viscous sublayer, however the length scale for the transverse y' coordinate is introduced differently. It is henceforth not required to know the exact relationship between the scales in which the distances along the normal from the streamlined plate are measured in the different domains. As usual, M_∞ denotes the free-stream uniform flow Mach number. The upper sign in front of the first term in the left side of (2.1) is taken for $M_\infty>1$ and the lower sign for $M_\infty<1$. After having solved (2.1), the vertical component v' of the velocity vector is determined from the relationship

$$v'=-|M_\infty^2-1|^{1/2}\frac{\partial}{\partial y'}\int\limits_{-\infty}^{x}p'(t,\xi,y',z)\,d\xi \tag{2.2}$$

when the perturbations damp out upstream at infinity, or by an equivalent formula for oscillations periodic in x.

The dependence of the desired gas parameters on the time and coordinates measured along the plate surface should be conserved in the potential domain exactly as was selected for the viscous near-wall sublayer. Namely

$$p'=aP'\exp(\omega t+kx+lz)$$

After substituting this into (2.1), we obtain an equation whose solution is

$$P'=\exp(my'),\quad m=i\,[\,|M_\infty^2-1|^{-1}l^2\mp k^2]^{1/2} \tag{2.3}$$

where the square root is understood to be the branch with $\mathrm{Re}\,m<0$. This latter requirement assures perturbation attentuation as $y'\to\infty$. However, a dependence between the wave numbers k and l can be indicated when $\mathrm{Re}\,m=0$ for both branches of the root. Such a situation will not be examined later.

By using the formula (2.2) for the transverse velocity vector component, we finally have

$$p'=a\exp(\omega t+kx+my'+lz) \tag{2.4}$$

$$v'=-a|M_\infty^2-1|^{1/2}\frac{m}{k}\exp(\omega t+kx+my'+lz)$$

We find the function $A(t, x, z)$ on which depends the magnitude of the instantaneous displacement of the streamline in the fundamental boundary-layer thickness that separates the thin near-wall region from the outer potential flow. As is shown in /1—4/

$$dA/dx = -|M_\infty^2 - 1|^{-1/2} v'(t, x, 0, z)$$

from which we conclude that

$$A = a\frac{m}{k^2}\exp(\omega t + kx + lz) \tag{2.5}$$

3. Dispersion relation.

Let us combine the solution for the viscous sublayer with the solution giving the gas parameters in the external domain. The combine conditions are in /1—4/: for $y \to \infty$

$$p \to p'(t, x, 0, z), \quad u - y \to A + \frac{1}{y}\frac{\partial^2}{\partial z^2}\int\limits_{-\infty}^{x} d\xi \int\limits_{-\infty}^{\xi} p'(t, \xi', 0, z)\, d\xi', \quad w \to -\frac{1}{y}\frac{\partial}{\partial z}\int\limits_{-\infty}^{x} p'(t, \xi, 0, z)\, d\xi \tag{3.1}$$

It is seen that joining the excess pressures in both solutions set up by (1.3) and (2.4), occurs automatically. Let us turn to the second condition in (3.1) by initially neglecting the transverse member in the right side that damps out as $1/y$ as $y \to \infty$. On the basis of (2.5), we find for the function A

$$df/dy \to -m/k^2 \text{ when } y \to \infty \tag{3.2}$$

Because of the third condition in (3.1), the side component of the velocity vector $w \to 0$ as $y \to \infty$. Hence

$$dh/dy \to 0 \text{ when } y \to \infty \tag{3.3}$$

Recalling the definition (1.5) of the function F and combining the limit relationships (3.2) and (3.3), we conclude that

$$\frac{dF}{dZ} \to -\frac{m}{k^{1/3}(k^2 + l^2)} \text{ when } |Z| \to \infty \tag{3.4}$$

As already noted above, the equation for F agrees with the equation describing waves propagation in plane-parallel flows. Hence, the solution (1.7) does not differ in form from that to which the nonstationary two-dimensional free boundary-layer interaction process is subject. However, the limit condition (3.4) contains a dependence on both the free-stream Mach number and on both wave numbers k and l. It is understandably desirable to convert it in such a way that it would agree with the analogous condition from the theory of the plane-parallel boundary layer.

In deriving the dispersion relation, we limit ourselves to just subsonic flows with $M_\infty < 1$. It is then natural to consider that the wave numbers k and l are pure imaginaries, and

$$m = -[|k|^2 + |M_\infty^2 - 1|^{-1}|l|^2]^{1/2}$$

where the square root is understood to be the arithmetic value. We set

$$k = \left(\frac{|k|m}{k^2 + l^2}\right)^{3/4} K, \quad \frac{k^2 + l^2}{m} = \mp i\left(\frac{|k|m}{k^2 + l^2}\right)^{-1/4} K \tag{3.5}$$

Both these equalities define a new parameter identically (the reduced wave number)

$$K = \pm i|k|^{1/4}\left(\frac{k^2 + l^2}{m}\right)^{3/4} \tag{3.6}$$

The upper sign in (3.5) and (3.6) is taken for $\operatorname{Im} k > 0$, and the lower for $\operatorname{Im} k < 0$. Since the ratio $(k^2 + l^2)/m$ is positive, the arguments k and K agree. The limit condition (3.4) takes the required form

$$dF/dZ \to \mp iK^{-4/3} \text{ when } |Z| \to \infty \tag{3.7}$$

The constant ζ in the solution (1.7) changes during transformation of the wave numbers. In order to retain its form invariant, we supplement (3.5) and (3.6) by the definition of the reduced frequency

$$\omega = \left(\frac{|k|m}{k^2 + l^2}\right)^{1/2}\Omega \tag{3.8}$$

where the arguments ω and Ω are identical. Then, in fact, $\zeta = \omega/k^{2/3} = \Omega/K^{2/3}$. By satisfying the boundary condition (3.7), we arrive at the dispersion relation

$$\frac{d\,\mathrm{Ai}(\zeta)}{dZ}\left[\int\limits_{\zeta}^{\infty}\mathrm{Ai}(Z)\,dZ\right]^{-1}=\mp iK^{4/3},\quad \zeta=\frac{\Omega}{K^{2/3}} \tag{3.9}$$

in the standard form for the theory of plane-parallel flows. It is seen from the expression (1.7) for the complex variable Z that the selection made above of the branch $k^{4/3}$ of the three-valued function will assure convergence of the improper integral in the left side of (3.9) after the passage to the new parameters.

If $M_\infty = 0$, which corresponds to an incompressible fluid, the root is $m = -(|k|^2 + |l|^2)^{1/2}$, from which

$$\omega=\frac{|k|^{1/2}}{(|k|^2+|l|^2)^{1/4}}\Omega,\quad k=\frac{|k|^{3/4}}{(|k|^2+|l|^2)^{3/8}}K \tag{3.10}$$

4. Stability criterion. The dispersion relation (3.9) possesses an uncountable set of roots. In order to see this, we use the results elucidated in /7/. The asymptotic expansion of the Airey function in the neighborhood of the real negative half-axis as $|Z| \to\infty$ can be written as

$$\mathrm{Ai}(Z)=\frac{1}{2\sqrt{\pi}}\left[\exp\left(-\frac{2}{3}Z^{3/2}\right)\sum_{m=0}^{\infty}c_mZ^{-3/2m-1/4}+i\exp\left(\frac{2}{3}Z^{3/2}\right)\sum_{m=0}^{\infty}(-1)^mc_mZ^{-3/2m-1/4}\right. \tag{4.1}$$

$$c_m=(-1)^m\frac{1}{(2m)!\,72^m}\prod_{n=1}^{3m}(2n-1),\quad c_0=1$$

where the regular branches of the multivalued functions are extracted by using the inequalities $\pi/3<\arg Z<5\pi/3$ of the branches of the argument of the complex variable Z. Let us put $\theta=\arg\zeta=\pi+\theta'$, and let us introduce the parameter $\chi=\theta'|\zeta|^{3/2}$. Let $\theta'\ll1$ and $\chi\ll1$ while $|\zeta|\gg1$. We now use the asymptotic formula (4.1) in which we retain only the principal terms, to simplify the dispersion relation. Summarizing, we find

$$|\zeta|^{1/4}\cos\left(\frac{2}{3}|\zeta|^{3/2}+\frac{\pi}{4}\right)=-\frac{\sqrt{3\pi}}{2}|K|^{4/3} \tag{4.2}$$

$$\theta'|\zeta|^{7/4}\sin\left(\frac{2}{3}|\zeta|^{3/2}+\frac{\pi}{4}\right)=\pm\frac{\sqrt{\pi}}{2}|K|^{4/3}$$

The upper sign is taken here for $\mathrm{Im}\,K>0$, and the lower for $\mathrm{Im}K<0$. The first of the equalities (4.2) is to determine $|\zeta|$. For finite values of $|K|$ and $|\zeta|\to\infty$ we approximately have

$$|\zeta_j|=\left[\frac{3}{2}\pi\left(j+\frac{1}{4}\right)\right]^{2/3}$$

This formula can understandably be used only for sufficiently large j. From the second relationship in (4.2) we find

$$\theta'=\pm(-1)^j(\sqrt{\pi}/2)|K|^{4/3}\left[\frac{3}{2}\pi\left(j+\frac{1}{4}\right)\right]^{-7/6}$$

Therefore, an uncountable set of eigenvalues $\mathrm{Re}\,\zeta_j$, $\mathrm{Im}\,\zeta_j$ correspond to each value of $|K|$, from which $\mathrm{Re}\,\Omega_j$, $\mathrm{Im}\,\Omega_j$ are computed. Is is convenient to give the wave numbers k, l in the calculation, and to determine thereby the magnitude of the root m from the second equality in (2.3). Formula (3.6) is for finding the reduced wave number K. Having obtained the value of Ω_j we can return to the original frequency ω_j by using the relationship (3.8).

Since $\arg K=\pm\pi/2$, and $\arg\zeta$ is close to π, then $\arg\Omega_j$ will differ slightly from $\pi\pm\pi/3$. But $\arg\omega_j=\arg\Omega_j$, hence $\mathrm{Re}\,\omega_j<0$ for sufficiently large j. The main deduction of the asymptotic analysis of the roots of the dispersion equation (3.9) is the following: free interaction of the internal waves being propagated in the boundary layer with a subsonic external flow is stable if $|\zeta_j|\gg1$. For a supersonic free stream velocity this conclusion is valid for all modes, including those appropriate to finite values of $|\zeta_j|$. The recent investigation /8/ showed, however, that the first mode of two-dimensional perturbations in a freely interacting incompressible boundary layer can turn out to be both stable and unstable, depending on the magnitude of the wave number k. Since the dispersion equation (3.9) does not contain the Mach number M_∞ explicitly, then for any free-stream subsonic velocity its first root ζ_1 will yield not only positive but also negative values of $\mathrm{Re}\,\omega_1$ for variations of both wave numbers k, l, prescribing the form of the spatial perturbations.

The dependences of the real and imaginary parts of the reduced frequency Ω on the absolute value of the reduced wave number K for the first three roots of the dispersion equation (3.9) are shown in Figs.1 and 2 (curves $1-3$, respectively). The dashed lines $1', 2', 3'$ are superposed on these same figures to compare the results of solving the dispersion equation in application to the supersonic boundary layer (in this case the right side of (3.9) $Q = \mp iK^{4/3}$ should be replaced by $Q = -K^{4/3}$, where $K = k$, $\Omega = \omega$). The behavior of the curves in Fig.1 confirms the deduction formulated above about the instability of the process of free first-mode interaction with the subsonic external flow if $|K| > K_*$. According to the calculations performed, the number corresponding to the neutral oscillations is $K_* = 1.0005$.

It is appropriate to recall now that all the considerations elucidated above were in the special dimensionless system of units. Going from the frequency ω and the wave numbers k and l in this system over to the frequency ν and the wave numbers α, γ in the original (also dimensionless) system of units is accomplished by means of the formulas /1—4/

$$\omega = C^{1/4}\lambda^{-3/2}N_{\mathrm{Re}}^{-1/4}|M_\infty^2 - 1|^{-1/4}\frac{T_w}{T_\infty}\nu \tag{4.3}$$
$$k = C^{3/8}\lambda^{-5/4}N_{\mathrm{Re}}^{-3/8}|M_\infty^2 - 1|^{-3/8}\left(\frac{T_w}{T_\infty}\right)^{3/2}\alpha$$
$$l = C^{3/8}\lambda^{-5/4}N_{\mathrm{Re}}^{-3/8}|M_\infty^2 - 1|^{-3/8}\left(\frac{T_w}{T_\infty}\right)^{3/2}\gamma$$

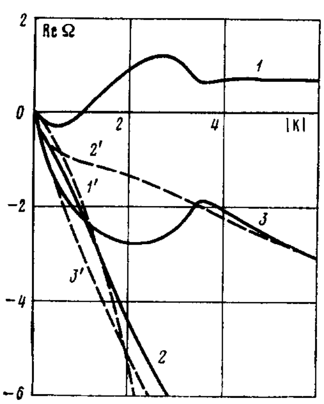

Fig.1

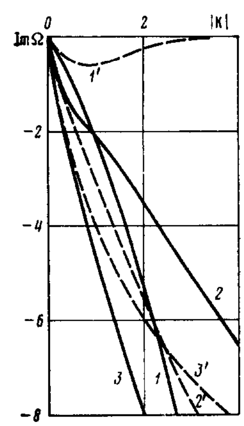

Fig.2

Here N_{Re} denotes the Reynolds number, C is the constant from the linear Chapman law for the viscosity and heat conduction coefficients, and T_w and T_∞ are the temperatures of the plate and the free stream, respectively. The constant λ is determined by using the equality $dU_2(0)/dy_2 = \lambda C^{-1/2}(T_w/T_\infty)^{-1}$ in which the function U_2 gives the distribution of the longitudinal component of the velocity vector in the unperturbed boundary layer as a function of its transverse coordinates y_2. According to the Blasius solution $\lambda = 0.3321$.

Subsitution of the formulas (4.3) into the dispersion relation (3.9) permits writing an expression for the constant ζ as

$$\zeta = \lambda^{-3/2}\frac{\nu}{\alpha^{1/2}} \tag{4.4}$$

and converting its right side Q to the form

$$Q = \exp\left(\pm\frac{\pi}{6}i\right)C^{1/2}\lambda^{-5/2}N_{\mathrm{Re}}^{-1/2}|M_\infty^2 - 1|^{-1/2}\cdot(T_w/T_\infty)^2|\alpha|^{1/2}(|\alpha|^2 + |\gamma|^2)/|\mu| \tag{4.5}$$
$$|\mu| = [\,|\alpha|^2 + |M_\infty^2 - 1|^{-1}|\gamma|^2]^{1/2}$$

As is clear from the above, the stability condition for a freely interacting subsonic boundary layer on a plate is

$$C^{1/2}\lambda^{-5/2}N_{\mathrm{Re}}^{-1/2}|M_\infty^2 - 1|^{-1/2}(T_w/T_\infty)^2|\alpha|^{1/2}(|\alpha|^2 + |\gamma|^2)/|\mu| \leqslant K_*^{4/3} \tag{4.6}$$

Here the equality sign corresponds to neutral oscillations whose amplitude remains invariant with time.

The criterion (4.6) permits making definite conclusions about the position of the neutral curve in the plane $N_{\mathrm{Re}}, |\alpha|$ which is ordinarily used in linear stability theory. An increase in the subsonic free stream Mach number shifts this curve downward if the Reynolds numbers is fixed, where the influence of the Mach number diminishes for three-dimensional perturbations. Cooling the plate leads to the opposite result: the neutral stability curve is shifted upward. Heating the streamlines surface is equivalent in action to increasing the Mach number at infinity. Superposition of oscillations in the side direction will diminish the critical values of $|\alpha|$ for which the perturbation amplitude is kept constant in time; for large values of $|\gamma|$ the change in Mach number has slight effect on the location of the neutral curve. Moreover, this curve can be shifted downward or upward depending on the magnitudes of the constants C and λ, in particular, diminution of the parameter λ causes diminution in the critical value of $|\alpha|$ for a given Reynolds number.

5. The function h. There remains to investigate the solution of the second equation from the system (1.4), to confirm the validity of the deductions made above. Using the notation $dh/dy = H$, we write it in the form of an inhomogeneous Airey equation

$$d^2H/dZ^2 - ZH = -l/k^{2/3} \tag{5.1}$$

on the plane of the complex variable Z. Application of the results elucidated in /7/ yields the following expression for the second linearly independent solution $\mathrm{Bi}(Z)$ of the Airey equation

$$\mathrm{Bi}(Z) = \exp\left(\frac{\pi}{6}i\right)\mathrm{Ai}\left(Z\exp\left(\frac{2\pi}{3}i\right)\right) + \exp\left(-\frac{\pi}{6}i\right)\mathrm{Ai}\left(Z\exp\left(-\frac{2\pi}{3}i\right)\right) \tag{5.2}$$

The behavior of the function $\mathrm{Bi}(Z)$ in different sectors of the plane Z as $|Z|\to\infty$ is established by using the asymptotic expansion (4.1) of the Airey function. It can be verified that in the sector extracted by the inequalities $-\pi/3 < \arg Z < \pi/3$

$$\mathrm{Bi}(Z) = \frac{1}{\sqrt{\pi}}\exp\left(\frac{2}{3}Z^{3/2}\right)\sum_{m=0}^{\infty}(-1)^m c_m Z^{-3/2m-1/4} \tag{5.3}$$

Let us first evaluate the Wronskian W of the solutions $\mathrm{Ai}(Z)$ and $\mathrm{Bi}(Z)$. According to the general theory of ordinary differential equations $W = \mathrm{const}$. Application of the expression (5.2) to find the numerical value of the constant turns out to have slight effect, consequently, we turn to the asymptotic formulas (4.1) and (5.3). We consequently find $W = 1/\pi$. The general solution of (5.1) can now be represented as

$$H = a_1\mathrm{Ai}(Z) + a_2\mathrm{Bi}(Z) + \frac{\pi l}{k^{2/3}}\left[\mathrm{Ai}(Z)\int_{\zeta}^{Z}\mathrm{Bi}(Z')\,dZ' - \mathrm{Bi}(Z)\int_{\zeta}^{Z}\mathrm{Ai}(Z')\,dZ'\right] \tag{5.4}$$

By using the asymptotic expansion of the functions $\mathrm{Ai}(Z)$ and $\mathrm{Bi}(Z)$ in the sector $-\pi/3 < \arg Z < \pi/3$, we establish the behavior of H as $|Z|\to\infty$. In a first approximation, an estimate of the integral terms contained in the square brackets in the right side of (5.4) yields

$$\mathrm{Ai}(Z)\int_{\zeta}^{Z}\mathrm{Bi}(Z')\,dZ' = \frac{1}{2\pi}\frac{1}{Z} + \cdots \tag{5.5}$$

$$\mathrm{Bi}(Z)\int_{\zeta}^{Z}\mathrm{Ai}(Z')\,dZ' = \frac{1}{\sqrt{\pi}}\frac{1}{Z^{1/4}}\exp\left(\frac{2}{3}Z^{3/2}\right)\int_{\zeta}^{\infty}\mathrm{Ai}(Z')\,dZ' - \frac{1}{2\pi}\frac{1}{Z} + \cdots$$

In order to eliminate exponential growth of H as $|Z|\to\infty$, it is necessary to set

$$a_2 = \frac{\pi l}{k^{2/3}}\int_{\zeta}^{\infty}\mathrm{Ai}(Z)\,dZ$$

From the adhesion condition on the streamlined surface: $H=0$ for $Z=\zeta$, there follows

$$a_1 = -\frac{\pi l}{k^{2/3}}\mathrm{Bi}(\zeta)[\mathrm{Ai}(\zeta)]^{-1}\int_{\zeta}^{\infty}\mathrm{Ai}(Z)\,dZ$$

Combining the results obtained, we have the expression

$$H = \frac{\pi l}{k^{2/3}}\left\{\mathrm{Bi}(Z)\int_{Z}^{\infty}\mathrm{Ai}(Z')\,dZ' + \mathrm{Ai}(Z)\left[\int_{\zeta}^{Z}\mathrm{Bi}(Z')\,dZ' - \frac{\mathrm{Bi}(\zeta)}{\mathrm{Ai}(\zeta)}\int_{\zeta}^{\infty}\mathrm{Ai}(Z')\,dZ'\right]\right\}$$

It retains its meaning for all ζ since the zeroes of the Airey function situated on the negative real half-axis do not agree with the roots of the dispersion relation (3.9). According to the asymptotic estimates (5.5), as $|Z|\to\infty$

$$H \to \frac{l}{k^{3/4}} \frac{1}{Z} + \dots \tag{5.6}$$

We now turn to the third condition in (3.1) which was satisfied only roughly above in the form (3.3). More accurate calculations show that $dh/dy \to l/(ky)$ as $y \to \infty$. This relation written on the plane of the complex variable Z agrees exactly with (5.6). It can be seen that taking account of the correction term proportional to $1/y$ in the second condition in (3.1), which had earlier been neglected for simplicity, does not change the form of (3.4) if the asymptotic representation (5.6) is taken into account for the function H. Therefore, the investigation of the second equation from the system (1.4) permits more accurate compliance with the conditions for combining the solutions for the viscous near-wall sublayer and the external potential flow.

6. Boundary layer in an incompressible fluid. In conclusion, we discuss briefly the results of linear stability theory in application to the boundary layer on a flat plate streamlines by an incompressible fluid.

Formulation of the general problem about flow stability and the main results of its investigation are elucidated in the books /5,6/. A description is contained therein, of the stability curves in the plane $N_{\mathrm{Re}}, |\alpha|$ determined by different, including numerical, methods, and a comparison is given between the data of theory and experiment. A detailed analysis, recently undertaken which includes taking account of several correction factors, permitted to establish the form of the lower branch of the neutral curve /9/ with great accuracy, where a three-layer flow model was used in this analysis which had been taken in the theory of free interaction.

The stability analysis is based on the Navier—Stokes equations and not on the simpler Prandtl equations. In reducing the Navier—Stokes equations to dimensionless form we select the distance from the nose of the plate, the free-stream velocity, and the fluid density as the fundamental measurement units. In this dimensionless system of units, the solution for the components u°, v° and w° of the velocity vector, and the pressure p° as a function of the time t° and the Cartesian coordinates $x^\circ, y^\circ, z^\circ$ are represented as

$$\{u^\circ - U_0, v^\circ, w^\circ, p^\circ - P_0\} = \{u_1, v_1, w_1, p_1\} \exp(\nu t^\circ + \alpha x^\circ + \gamma z^\circ) \tag{6.1}$$

Here the function U_0 yields a distribution of the longitudinal velocity vector component in the unperturbed flow along the transverse coordinate y°, the value of the constant P_0 is determined by the pressure in the free-stream, the complex amplitudes u_1, v_1, w_1 and p_1 satisfy the ordinary differential equations obtained by linearization of the Navier—Stokes equations. The relationships between the frequencies ω and ν and the wave numbers k, l and α, γ are established by (4.3) which reduce in the case under consideration to

$$\omega = \lambda^{-3/2} N_{\mathrm{Re}}^{-1/4} \nu, \quad k = \lambda^{-5/4} N_{\mathrm{Re}} \alpha, \quad l = \lambda^{-5/4} N_{\mathrm{Re}}^{-3/8} \gamma \tag{6.2}$$

Let us consider the equation for the complex amplitudes. Following Squire, we put /5,6/

$$v_1 = \mu F^\circ, \quad \mu \frac{dF^\circ}{dy^\circ} = -(\alpha u_1 + \gamma w_1), \quad \mu^2 = \alpha^2 + \gamma^2 = -|\mu|^2 \tag{6.3}$$

and use the notation $c = -\nu/\alpha$ for the phase velocity of the wave. The continuity equation is here satisfied identically, and the rest are converted as follows:

$$(U_0 - c)\left(\frac{d^2F^\circ}{dy^{\circ 2}} + \mu^2 F^\circ\right) - \frac{d^2U_0}{dy^{\circ 2}} F^\circ = \frac{1}{\mu N_{\mathrm{Re}}^*}\left(\frac{d^4F^\circ}{dy^{\circ 4}} + 2\mu^2 \frac{d^2F^\circ}{dy^{\circ 2}} + \mu^4 F^\circ\right) \tag{6.4}$$

$$p^* = (U_0 - c)\frac{dF^\circ}{dy^\circ} - \frac{dU_0}{dy^\circ} F^\circ - \frac{1}{\mu N_{\mathrm{Re}}^*}\left(\mu^2 \frac{dF^\circ}{dy^\circ} + \frac{d^3F^\circ}{dy^{\circ 3}}\right)$$

The first of the equations written is the Orr—Sommerfeld equation for the function F° with the sole difference that the modified Reynolds number is $N_{\mathrm{Re}}^* = \alpha N_{\mathrm{Re}}/\mu$. The second equation in (6.4) is to evaluate $p_1 = \alpha p^*/\mu$. Therefore, the problem of spatial waves being propagated in an incompressible boundary layer on a plate will reduce to an analogous problem about perturbed plane-parallel flow /5,6/. An analogous conclusion was formulated above within the framework of the theory of free interaction, where (1.5) and (6.3) by means of which the new desired functions F and F° were introduced, differ only by a nonessential scale factor. Furthermore, as is shown in /8/, the solution of the problem of stability of the incompressible boundary layer relative to two-dimensional perturbations agrees completely with the solution predicted by the theory of free interaction if $|\alpha| \to \infty$, $|\alpha|/\sqrt{N_{Re}} \to 0$ and $N_{Re} \to \infty$. It hence follows that the deduction about what is the nature of three-dimensional perturbation development in the boundary layer on a plate can be given a basis by a study of its free interaction with an external potential flow under the conditions that $|\mu| \to 0$, $N_{Re}^* \to \infty$ and the ratio $|\alpha|/|\mu|$ is on the order of magnitude of one. No special investigation of (6.4) is required since all the results needed are in /8,9/.

Within the boundary layer the function is $U_0(y^\circ) = U_2(y_2)$. The passage from this to the modified Reynolds number denotes an affine stretching of the coordinate y_2 in conformity with the equality $y_2 = \sqrt{\mu/\alpha}\, y_2^*$. Since $U_2(y_2) = U_2^*(y_2^*)$, the phase velocity of the wave must be kept invariant. This requirement implies the expression $\nu = \alpha\nu^*/\mu$ for the three-dimensional perturbations; indeed $c = -\nu/\alpha = -\nu^*/\mu$. Now turning to /8,9/, we at once write down the dispersion relation

$$\frac{d\,\mathrm{Ai}(\zeta)}{dZ}\left[\int_\zeta^\infty \mathrm{Ai}(Z)\,dZ\right]^{-1} = \mp i\lambda^{*-5/4} N_{Re}^{*-1/2} \mu^{4/3}, \quad \zeta = \lambda^{*-2/3}\frac{\nu^*}{\mu^{2/3}} \tag{6.5}$$

where the upper sign is taken for $\mathrm{Im}\,\mu > 0$ and the lower for $\mathrm{Im}\,\mu < 0$, the constant is $\lambda = \sqrt{\alpha/\mu}\,\lambda^*$ because of the equality $\lambda y_2 = \lambda^* y_2^*$. It is seen that the expression for ζ in (6.5) agrees with that in (4.4). The velocity field for incompressible fluid flow is independent of the parameters C and T_w/T_∞, and the Mach number is $M_\infty = 0$. Under these conditions the right side Q of the dispersion relation under consideration goes over into (4.5). The relationship to the frequency ω and the wave numbers k, l that were used to study the free boundary-layer interaction with the external potential flow is realized by using (6.2). The canonical form of the dispersion relation (3.9) is obtained for parameters Ω and K introduced by (3.10).

In a narrow, near-wall domain the solution (6.1) can be written in the form (1.3) if the normalizing constants are selected in a suitable manner /8/. Therefore, the internal waves being studied above in a freely interacting incompressible boundary layer are the asymptotic of the oblique Tollmien–Schlichting waves /5,6/, being propagated at an angle to the main stream direction, in the limit when $|\alpha| \to \infty$, $|\alpha|/\sqrt{N_{Re}} \to 0$, $N_{Re} \to \infty$ and $|\alpha|/|\mu| \sim 1$.

The analysis made permits formulation of a nonlinear problem on the stability of a freely interaction boundary layer in the subsonic gas flow with respect to spatial perturbations. To do this it is sufficient to require periodicity of the required functions in the variables x and z without involving linearization of (1.1). The external flow domain will be described, as before, by the linear equation (2.1). The experimental data indicate that the nonlinear stage of unstable Tollmien–Schlichting wave amplification terminates by their breaking up and going over into three-dimensional perturbations if even initially they are plane-parallel. As has been shown above, the imposition of oscillations in the side direction is accompanied by losses in the stability of the longer-wave perturbations for a fixed Reynolds number.

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